

DYNAMICS OF THREE VORTICES ON A PLANE AND A SPHERE — III

Noncompact case. Problems of collaps and scattering

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Abstract

In this article we considered the integrable problems of three vortices on a plane and sphere for noncompact case. We investigated explicitly the problems of a collapse and a scattering of vortices and obtained the conditions of its realization. We completed the bifurcation analysis and investigated for collinear and Thomson's configurations the dependence of stability in linear approximation and frequency of rotation in relative coordinates from value of a full moment. We indicated the geometric interpretation for characteristic situations. We constructed a phase portrait and geometric projection for the integrable configuration of four vortices on a plane.

1 Motion of vortices on a plane

Let us consider motion of vortices on a plane under the following condition (a_k are inverse intensities of point vortices $a_k = 1/\Gamma_k$)

$$A = A_1 a_2 + a_2 a_3 + a_1 a_3 \leq 0. \quad (1)$$

In case $A < 0$ algebra of Poisson brackets is the algebra $\mathbb{R} \oplus so(2, 1)$, and in case $A = 0$ it is solvable algebra. As indicated in [12], the symplectic sheet in both cases

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is noncompact (in the first case the symplectic sheet is hyperboloid, in second it is paraboloid), and the trajectories of a representing point on it can be both finite and infinite (this also concerns the dynamics of three vortices in relative distances). In the second case one can speak about a scattering of vortices.

Let us consider, at first, the case $A < 0$. Without loss of generality let us put $\Gamma_2, \Gamma_3 > 0$, $\Gamma_1 < 0$, $-\Gamma_1 < \Gamma_2 + \Gamma_3 > 0$. Further, D is the central linear function of algebras of Lie–Poisson brackets ($D = \sum a_k M_k$, M_k are quadrates of mutual distances between vortices). From

$$\lambda^2 = \frac{-3a_1 a_2 a_3}{D^2} (a_1 + a_2 + a_3)^3, \quad (2)$$

for a stability coefficient of Thomson’s configurations, it follows, that the stability of such configuration in a linear approximation for various D is determined by sign of the sum of inverse intensities $S = \sum a_k$. Therefore, we shall consider three separate cases corresponding to the values $S > 0$, $S < 0$, and $S = 0$.

The geometric interpretation and the bifurcation diagrams for various characteristic combinations of parameters A , S and D are indicated correspondingly in a Fig. 1 and Fig. 2.

Case $A < 0$ and $S > 0$. The geometric interpretation shows that in this case for $D \leq 0$ only finite motions are possible for which trajectory on a plane of an integral of the full moment is limited from both sides by the condition $\Delta^2 > 0$.

The bifurcation diagram (Fig. 2a) in this case reminds the diagram for vortices on a plane in compact case for positive intensities with the exclusion, that there is only one collinear configuration. Another difference from compact case is that the type of stability varies: Thomson’s configurations are stable for positive intensities in compact case and are unstable in this case (Fig. 3a), and the collinear configurations are stable. Both Thomson’s, and the collinear solutions are determined only in area $D > 0$ (Fig. 1c). For positive values of the full moment both finite, and infinite motions are possible. By virtue of instability of Thomson’s solution, the solutions with any positive energy are allowed.

Case $A < 0$ and $S < 0$. In this case for any value of the full moment the motion is finite (Fig. 1d–f). Thomson’s and the collinear solutions are determined only for $D < 0$. That solutions are remarkable because the energy of all configurations tends to infinity with decrease of the absolute value of the full moment down to zero (see Fig. 2b). The curve of Thomson’s configurations is situated above of the curve for collinear configurations and limits from above the area of possible movements. They are accordingly stable and unstable in a linear approximation (Fig. 3b).

Case $A < 0$ and $S = 0$. For the value $D < 0$ the tangency of the trajectory with the boundary of area is possible. This trajectory corresponds to stable (see Fig. 3c) collinear configuration (Fig. 2c), which energy tends to zero with decrease of D . Thomson’s solutions are presented only for $D = 0$ and are degenerated, and

have a neutral equilibrium ($\lambda^2 = 0$). Such solutions are possible for any distances between vortices and fill the whole straight line at Fig. 1h. The dependence of an angular velocity from the distance between vortices is determined by the formula [10]

$$\Omega = \frac{(\Gamma_1 + \Gamma_2 + \Gamma_3)}{2\pi M}, \quad (3)$$

where M is quadrate of the distance between vortices. This straight line, corresponding to Thomson's configurations, divides areas of scattering and collapsing behavior of three vortices. Under a "collapse" we understand the process of simultaneous collision of (in this case, three) vortices. More detailed study of emergence of a collapse is indicated further.

For $D \neq 0$ the finite motions are presented for any value of the moment, and infinite motions occurred only for negative values.

The dependence of an angular velocity of rotation both Thomson's and collinear configurations in first and second cases is qualitatively identical and is defined by functions monotonically decreasing, with increase of absolute value of a moment. For case $S = 0$ the angular velocity of rotation of a collinear configuration is slowly increased with increase of the absolute value D .

The last case corresponding to noncompact motion of vortices arises under condition of $A = 0$ (solvable algebra). The motion is possible only for positive values of D . There are only two stationary configurations: one Thomson's and one collinear (Fig. 2d). For $D = 0$ three vortices are placed on one straight line, and each vortex is placed at center of a vorticity of two remaining. It is a corollary of general possibility of reducing of a problem of $n + 1$ vortices to problem of n vortices.

2 Motion of vortices on a sphere

Similarly to Section 1 we shall construct the bifurcation diagrams for the sphere, and also plots of absolute motion, using for small D appropriate dependencies for a plane and the method of a continuation on the parameter. The values $d_k = 4R^2(a_i + a_j)$ determine those values of the full moment, which can arise in problem of two vortices. The geometric interpretation for motion on the sphere is indicated in a Fig. 5.

Case $A < 0$ and $S > 0$. In addition to Thomson's and collinear configurations that exists in plane case, for $D > 0$ there are two collinear configurations, appearing from a problem of two vortices (Fig. 4a). By continuation on the moment one of them merges with Thomson's, which disappears for value d_T . The collinear configuration continues and merges with another collinear configuration, that has an analog on the plane. After this both configurations disappears also. For $D > d_3$ and $D < d_1$ the stationary configurations do not exist. Thomson's configuration are in this case always unstable (Fig. 6a), and the collinear configurations are stable.

Case $A < 0$ and $S < 0$. In this case the confluence of Thomson's and one of collinear (closest to Thomson's on energy) configurations (Fig. 4b) are happened. Besides there are collinear configurations from a problem of two vortices (for one of them $E \mapsto \infty$ by coincidence of vortices because of existence of one negative intensity). One of them extended into area of positive values of D . Thomson's configuration, as well as in plane case, is stable in the linear approximation (Fig. 6b). Some of collinear configurations are also stable. One of them becomes stable in confluence with Thomson's configuration, and other loses stability in confluence with unstable collinear configuration.

Case $A < 0$ and $S = 0$. Instead of one collinear configuration in case of a plane in this case there are three various branches of collinear configurations, as for $D > 0$ and for $D < 0$ (see Fig. 4c). The solution with higher energy is unstable for $D > 0$ and disappears merging with collinear configuration with lower energy. The solution with the lowest energy is stable. For $D < 0$ situation is reversed. Thus collinear solution with lower energy merges with the solution, appeared at $D = d_2$. As well as on the plane, the Thomson solution are present only for $D = 0$. In this case all trajectories are collapsing (Fig. 5h).

At last, under condition of $A = 0$, the bifurcation diagram is indicated in a Fig. 4d. As well as in plane case, the stationary solutions exist only for $D > 0$, (Fig. 5 j-l). The difference consists in emerging of collinear configurations of a problem of two vortices. One such configuration merges with Thomson's, then both disappear. All collinear solutions are stable, and Thomson's is unstable (Fig. 6d).

The dependence of angular velocity of rotation for the listed situations is represented in a Fig. 7. As well as in compact case, the frequency of rotation increasing with creation of the third vortex from the problem of two vortices.

Remark 1. *Such magnification, as well as in compact case, is explained by the fact, that for the birth of a pair of vortices from one vortex of total intensity there is a rotation of a collinear configuration around an axis, which is passing through the third vortex. In the result this vortex actually does not influence on rotation of two appearing vortices, which angular velocity depends on the mutual distance between them as*

$$\Omega = \frac{1}{2\pi M_3} \sqrt{(\Gamma_1 + \Gamma_2)^2 - \frac{\Gamma_1 \Gamma_2 M_3}{R^2}}, \quad (4)$$

Where M_3 is quadrate of the distance between first two vortices, and Γ_1, Γ_2 are intensities, corresponding to it. As the last expression shows the angular velocity tends to infinity with $M \mapsto 0$.

The study of dynamics of angles of declination of rotation axis to the plane of vortices shows, that in first and second listed cases the angle of declination of axes increasing from zero up to $\pi/2$. So, therefore the rotation axis tends to take a position in the plane of vortices and appropriate Thomson's a configuration turned into collinear (Fig. 8).

Remark 2. *Note, that a condition of existence of static configurations on a sphere for a specific collection of intensities is*

$$a_1 a_2 a_3 (a_1 + a_2 + a_3) > 0, \quad (5)$$

which determines stability of Thomson's configuration

$$\lambda^2 = \frac{D - 3R^2(a_1 + a_2 + a_3)}{9D^2} a_1 a_2 a_3 (a_1 + a_2 + a_3). \quad (6)$$

This correlation is easy for understanding from the geometric interpretation (see Fig. 5d), which illustrates a possibility of reaching of function H to its extremum. Let us note, that as well as in compact case, that was considered in [12], the static configurations on a sphere are observed only under fulfillment of the condition $\Gamma_1 \Gamma_2 + \Gamma_2 \Gamma_3 + \Gamma_1 \Gamma_3 > 0$, (Fig. 7).

3 The condition of a collapse of vortices on a plane and a sphere

Let us consider conditions of emergence of a collapse of three vortices on a plane and a sphere. It is known, that the collapse is impossible for any pairs of vortices taken separately, because with the approach of such pair influence of the third (remote) vortex negligible small, and two vortices on a plane (sphere) move relatively each other so, that the distance between them did not change. The necessary condition of the collapse is the fulfillment of $D = 0$, since it follows $M_k = 0$. The condition $D = 0$ allows to proceed from four-dimensional to three-dimensional Lie algebra, for which actual dynamics takes place on the singular symplectic sheet, so that the nontrivial singular symplectic sheet (surface of a cone) will correspond only to algebra $so(2, 1)$. The similar statements are valid also for a simultaneous collapse n vortices, which is not almost investigated (except for cases $n = 4, 5$) [1, 10].

The conditions of a collapse of vortices on a sphere, obviously, will coincide with conditions for a plane, since on small distances the influence of curvature to dynamics of vortices is insignificant.

Remark 3. *The problem of the collapse is one of the most interesting problems connected to vortices and represents large interest for the theoretical hydromechanics as one of the models, which can be used for understanding of the transition to the turbulence contained in a nonuniqueness of solutions of the hydrodynamic equations of the Euler. Really, the theorem of existence and uniquenesses for these equations are proved in the supposition of sufficient smoothness of an initial field of velocities. From a mathematical point of view the process of the collapse of vortices representing confluence of the special solutions of Euler equation of δ -function type, under*

inverting of time will determine disintegration of vortices with corresponding loss of uniqueness. Therefore large interest represents the study of this problem from the point of view of a regularization of collisions in the similar way, how it is done in classical celestial mechanics [3]. For physics of atmosphere the phenomenon of the collapse can be considered as a model of forming of large atmospheric vortices.

We research at first possibility of a homogeneous collapse for three vortices. For the homogeneous collapse all distances between vortices depend on time in the identical way and are in the constant proportion [10]. Using isomorphism with a problem of Lottka–Volterra (both for a plane, and a sphere [11]), we shall consider a homogeneous system of equations in the form

$$\hookrightarrow \dot{M}_1 = \Gamma_1 M_1 (M_2 - M_3). \quad (7)$$

The homogeneous asymptotic solutions (7) we shall search as

$$M_k = \frac{C_k}{\tau}. \quad (8)$$

Let's note, that such solutions are used in Kovalevskaya method for construction of full-parametric Laurent expansion. The nontrivial solutions of such form are possible only under condition of $S = 0$. If this condition is fulfilled, the solution can be written as

$$M_1 = \frac{C}{\tau}, \quad M_2 = \frac{C + a_3}{\tau}, \quad M_3 = \frac{C - a_2}{\tau}, \quad (9)$$

where C is arbitrary constant.

The solution (9) is full-parametric and contains only collapsing and running up (for a plane) trajectories. However, there is still the special solution $M_1 = M_2 = M_3 = \text{const}$, describing the Thomson's configuration, which in this case are degenerate. Using the relation

$$dt = \frac{2\pi M_1 M_2 M_3}{\Delta} d\tau, \quad (10)$$

it is possible to receive an asymptotics of the solution (9) in real time

1. for a plane: $t = \frac{D}{\tau}, M_k = C_k^* t;$
2. for a sphere: $t = AR^2 \left(1 - \sqrt{1 - \frac{B}{\tau R^2}}\right), M_k = D_k R^2 \left(1 - \left(1 - \frac{t}{AR^2}\right)^2\right),$

where $A, B, C_k^*, D, D_k = \text{const}$, R is radius of a sphere. For a sphere the absolute motion of vortices for conditions $D = 0$ and $S = 0$ consists in scattering of vortices from one point up to the moment of reaching of equator and further approach in some other point. Let us analyze the possibility of the collapse in the system of three

vortices on a plane in general inhomogeneous case. Writing the necessary condition of the collapse $D = 0$ in absolute variables

$$\left(\sum \Gamma_k\right) I - P^2 - Q^2 = 0, \quad (11)$$

where the uninvolute integrals of motion P, Q, I are equal

$$P = \sum \Gamma_k x_k, \quad Q = \sum \Gamma_k y_k, \quad I = \sum \Gamma_k (x_k^2 + y_k^2), \quad (12)$$

we found two special cases $\sum \Gamma_k = 0$ and $\sum \Gamma_k \neq 0$. Without loss of generality let us assume $\Gamma_1 = -1, \Gamma_2, \Gamma_3 > 0$. Under conditions $\sum \Gamma_k = 0$, from (12) it follows that $P = Q = 0$. It means, that the third vortex is at center of vorticity of first two vortices, rotating uniformly with frequency

$$\Omega = \frac{1}{2\pi} \frac{\Gamma_1 + \Gamma_2}{M_3} \left(1 + \frac{\Gamma_1}{\Gamma_2} + \frac{\Gamma_2}{\Gamma_1}\right),$$

around the point with the radius-vector

$$\vec{r} = \frac{\Gamma'_1 \vec{r}_1 + \Gamma'_2 \vec{r}_2}{\Gamma'_1 + \Gamma'_2}, \quad \Gamma'_1 = \Gamma_1 - \frac{(\Gamma_1 + \Gamma_2)^2}{\Gamma_1}, \quad \Gamma'_2 = \Gamma_2 - \frac{(\Gamma_1 + \Gamma_2)^2}{\Gamma_2},$$

where M_3 is quadrate of the distance between the first and second vortices, and \vec{r}_k, Γ_k are the radius-vector and intensity correspondingly. Therefore, collapse for case of zero total intensity is impossible.

Under condition of $\sum \Gamma_k \neq 0$ the collapse is possible only under condition of noncompactness of algebra of vortices $A = a_1 a_2 - a_1 - a_2 < 0$, as in compact case a symplectic sheet (the sphere) does not pass through a beginning of coordinates $M_k = 0$.

For the determination of sufficient conditions of the collapse we shall consider the projection of trajectories on a plane M_1, M_2 . Let us express M_3 from the integral of the full moment (for $D = 0$)

$$M_3 = a_1 M_1 + a_2 M_2. \quad (13)$$

The physical area on a plane M_1, M_2 for $D = 0$ is limited by straight lines, passing through the zero with coefficient of declination

$$K_{1,2} = \left(\frac{1 \pm \sqrt{-A}}{1 - a_2} \right). \quad (14)$$

The trajectory of regularized system is determined by the relation (13) and equation for energy of the system, which can be presented as

$$M_1^{-a_1} M_2^{-a_2} (a_1 M_1 + a_2 M_2) = \text{const}(E). \quad (15)$$

The analysis of the equation (15) shows, that for various values of parameters a_1, a_2 there are three various kinds of trajectories:

1. if $a_1 + a_2 < 1$, then all trajectories are compact, looks like closed loops, which are going out the origin of coordinates, tangenting axes OM_1 and OM_2 (Fig. 9a, curves b, c);
2. if $a_1 + a_2 > 1$, then all trajectories are noncompact, going to infinity, asymptotically press oneself to axes OM_1 and OM_2 ;
3. if $a_1 + a_2 = 1$, then all trajectories are straight lines which are going out from the origin of coordinates under different angles (the Fig. 9a, curve a);

Let us mark on a plane of parameters a_1, a_2 , areas appropriate to given types of trajectories. It is necessary to mark on this plane also values of parameters describing various forms of physical area. By (15), the physical area for $a_1 \neq 1, a_2 \neq 1$ (such, that $A < 0$), represents an interior of an sharp angle located inside a quadrant $M_1 > 0, M_2 > 0$. If $a_1 = 1$ ($a_2 = 1$), one of the sides of physical area coincides with an axes OM_2 (OM_1). In case $a_1 = a_2 = 1$ the motion is allowed in the whole quadrant $M_1 > 0, M_2 > 0$.

Comparing possible types of trajectories (for $A < 0, D = 0$) with types of areas of motion with consideration, that on reaching of the boundary motion continues on the same trajectory in the opposite direction, we conclude:

1. In the system of three vortices the homogeneous collapse (scattering) is possible, it happens when the relation $a_1 + a_2 = 1$ is executed for inverse intensities;
2. Scattering of vortices is possible only under conditions $a_1 = 1, a_2 = 1, a_1 = a_2 = 1$ (in this case the vortex motion never passes through the collinear configuration);
3. For other values of inverse intensities a_i the motion of vortices is bounded between two collinear configurations, and the distance between them is limited (see Fig. 9b).

4 Scattering of vortices on a plane

In the system of three vortices we shall name scattering trajectories, such trajectories that at least one of mutual distances (M_1, M_2, M_3) is infinitely increased. In difference from the collapse, the scattering can happen for $D \neq 0$.

Let us define new variables, with the help of which scattering problem of vortices on a plane can be reduced to the research of collapse:

$$\hookrightarrow X_1 = \frac{1}{M_2 M_3}. \quad (16)$$

Their equations of motion have a form

$$\hookrightarrow \dot{x}_1 = x_1((\Gamma_2 - \Gamma_3)x_1 + \Gamma_3x_2 - \Gamma_2x_3), \quad (17)$$

and the bounding physical area relation $\Delta^2 \geq 0$, will not be changed

$$2(x_1x_2 + x_1x_3 + x_2x_3) - x_1^2 - x_2^2 - x_3^2 \geq 0. \quad (18)$$

Let us select, as above $a_3 = -1$, $a_1, a_2 > 0$. The trajectory in space of variables x_1, x_2, x_3 is set by integrals of the moment D and the energy E , which can be presented in the form

$$a_1x_1 + a_2x_2 + a_3x_3 = D\sqrt{x_1x_2x_3}, \quad (19)$$

$$x_1^{a_2-a_1-1}x_2^{a_1-a_2-1}x_1^{1+a_1+a_2} = C(E), \quad (20)$$

where $C(E)$ is some function of energy.

For $D = 0$ the type of trajectories, defined by (19), is similar to the one investigated above. In this case to scattering 2) in a system of three vortices corresponds an inhomogeneous collapse in a system (17), for conditions $a_1 = 1$, $a_2 = 1$, $a_1 = a_2 = 1$ (see Fig. 9b).

It is interesting to note, that the homogeneous collapse and the scattering in variables M_k remains homogeneous in variables x_k , only direction of motion on trajectories varies.

Obvious quadratures and the analysis of absolute motion is in this case indicated in [10].

The numerical research for $D \neq 0$ shows, that in variables M_k there are only trajectories of types 1), 3) of the previous section. In this connection, it seems, that the availability of vortex pairs ($a_1 = 1$, $a_2 = 1$, $a_1 = a_2 = 1$) in the system is necessary and sufficient condition of scattering.

Remark 4. *In a work [6] reducing of the order is executed and the phase portraits of integrable case of a problem with zero total intensity and zero total moment are indicated. In this work the canonical form of equations of motion is used. For the indicated conditions on a level of algebra of brackets there is an reduction of a problem of four vortices to a problem of three vortices with the reduced Hamiltonian. On a symplectic sheet of three-dimensional vortex algebra, defined by introduced (as a corollary of the integrability) an invariant relation such as an integral of the moment with a constant D_1 , standard canonical variables L, l are presented. The condition of compactness of a symplectic sheet will have a very simple form, if three vortices of four have intensities of same sign. In compact case, for various values of intensities the phase portraits are represented at Fig. 10. In a general situation of unequal intensities there are six collinear stable solutions and three uncollinear*

unstable ones. The last solutions generalize of Thomson's configuration, however distances between vortices are not equal. The connection between energy E and moment D_1 for these solutions is determined by relation

$$E = F(\Gamma_1, \Gamma_2, \Gamma_3) D_1^{\frac{1}{8\pi}(\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2)},$$

where $f(\Gamma_1, \Gamma_2, \Gamma_3)$ is some function dependent on intensities. The bifurcation analysis, consisting in a determination of an explicit form of function $f(\Gamma_1, \Gamma_2, \Gamma_3)$, can be executed similarly to Section 1. In case if there are only two intensities of the same sign, the symplectic sheet is noncompact and the scattering is possible. The regular scattering, for example, is possible in case of interaction two (generally speaking, various) vortex pairs [4]. As far as we know, in general case the conditions of a collapse and a scattering in considered problem are not investigated.

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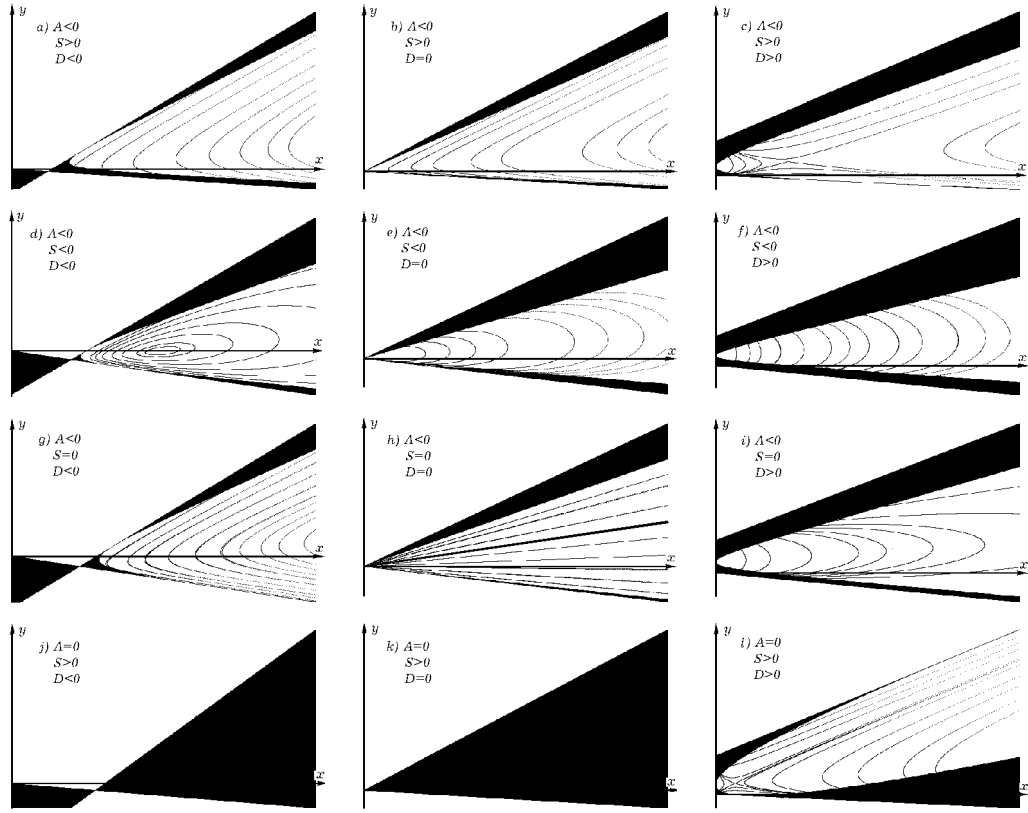


Figure 1: The geometric interpretation for various values of parameters A , S , D . The dark color designates area of positive values $M_k \geq 0$, for which $\Delta^2 > 0$. For figure h) the bold line corresponds to a set of Thomson's configurations.

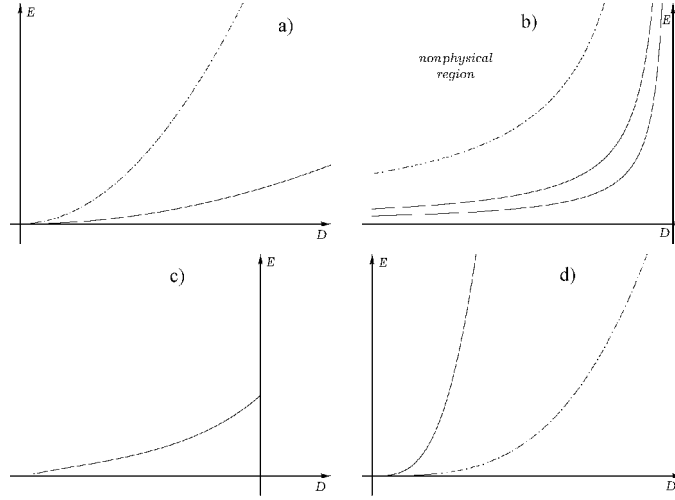


Figure 2: Bifurcation curves on a plane for cases a) $A < 0$ and $S > 0$; b) $A < 0$ and $S < 0$; c) $A < 0$ and $S = 0$; d) $A = 0$ and $S > 0$. Here and further dot-dashed line corresponds to Thomson's configurations, and solid lines to collinear.

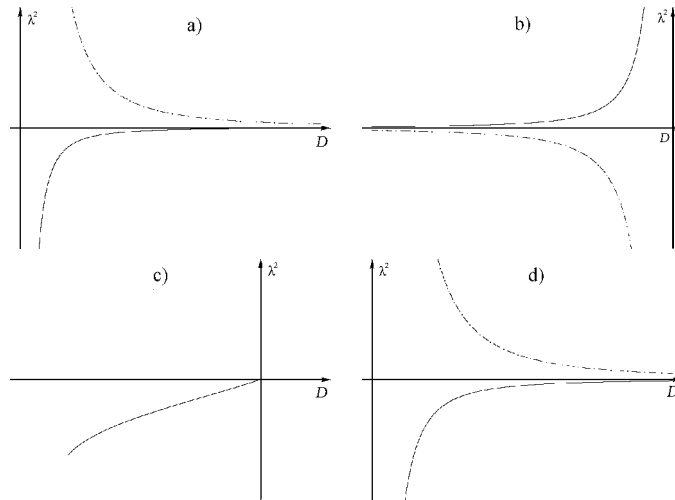


Figure 3: The stability coefficient on a plane for cases a) $A < 0$ and $S > 0$; b) $A < 0$ and $S < 0$; c) $A < 0$ and $S = 0$; d) $A = 0$ and $S > 0$.

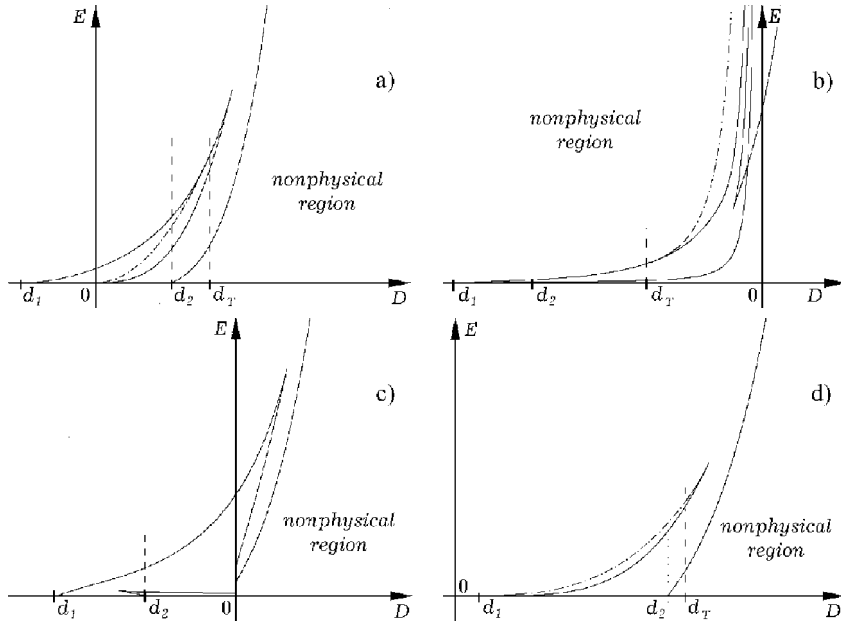


Figure 4: Bifurcation curves on a sphere for cases a) $A < 0$ and $S > 0$; b) $A < 0$ and $S < 0$; c) $A < 0$ and $S = 0$; d) $A = 0$ and $S > 0$.

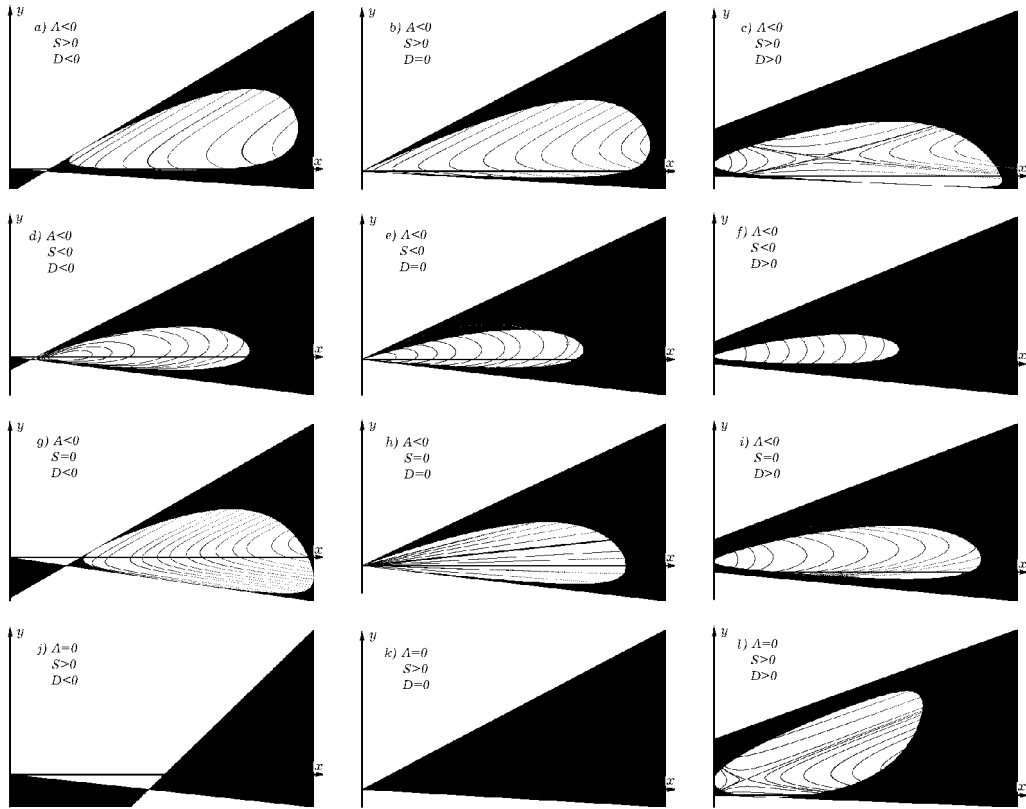


Figure 5: The geometric interpretation for a sphere for various values of parameters A , S , D . The dark color designates area of positive values $M_k \geq 0$, which for $\Delta^2 > 0$.

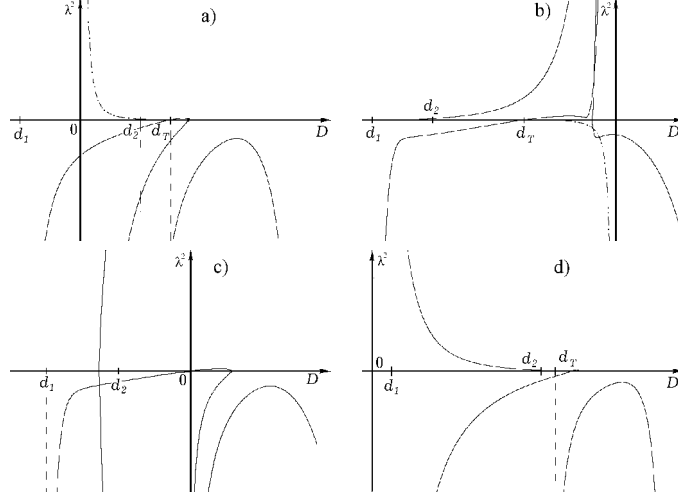


Figure 6: Stability coefficient on a sphere for cases a) $A < 0$ and $S > 0$; b) $A < 0$ and $S < 0$; c) $A < 0$ and $S = 0$; d) $A = 0$ and $S > 0$.

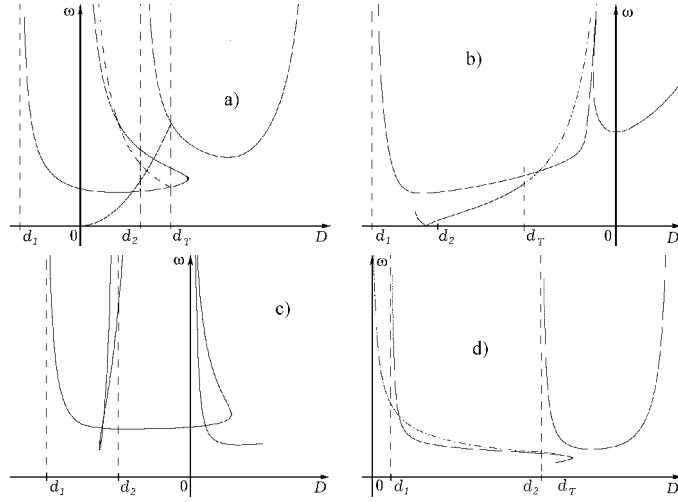


Figure 7: An angular velocity on a sphere for cases a) $A < 0$ and $S > 0$; b) $A < 0$ and $S < 0$; c) $A < 0$ and $S = 0$; d) $A = 0$ and $S > 0$.

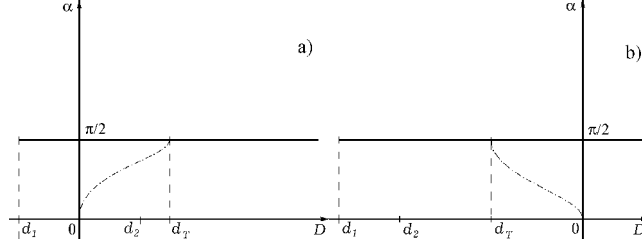


Figure 8: The angle of declination on a sphere for cases a) $A < 0$ and $S > 0$; b) $A < 0$ and $S < 0$.

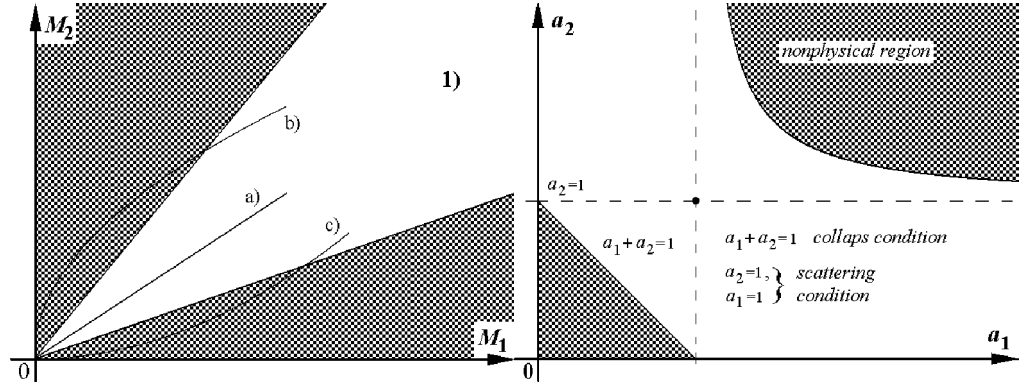


Figure 9: 1) Possible asymptotics of a behavior of trajectories of the system of three vortices near to zero on a plane (M_1, M_2) in case of the collapse (the nonphysical area is shaded); 2) Planes of parameters (a_1, a_2) for $a_3 = 1$. The collapse is possible on straight line $a_2 = a_1 - 1$, scattering corresponds to asymptotes $a_1 = 1, a_2 = 1$.

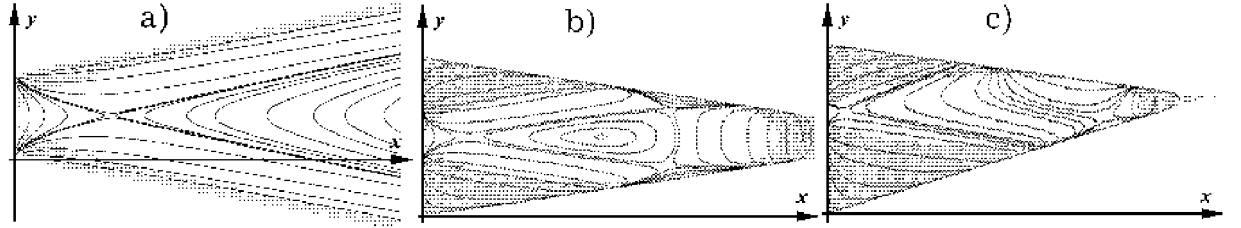


Figure 10: The geometric projection for case of four vortices on a plane with zero full intensity for a) $a_1 = -a_2 = a_3 = -a_4$; b) $a_1 = a_2 = a_3 \neq a_4$; c) $a_1 \neq a_2 \neq a_3$.

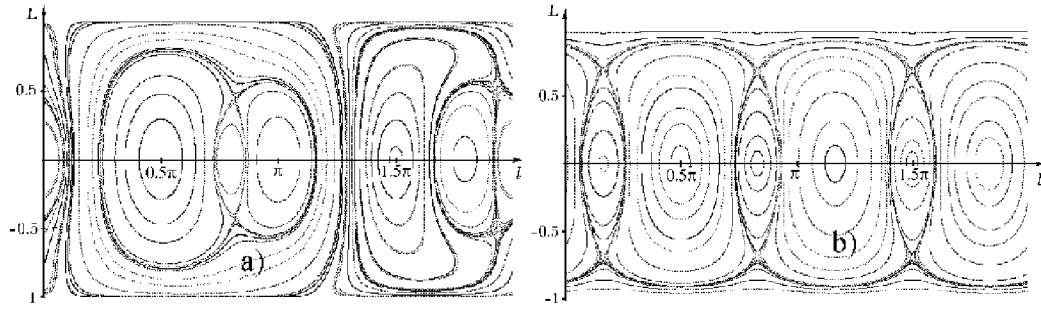


Figure 11: The phase portrait of motion of four vortices on a plane with zero total intensity for a) $a_1 \neq a_2 \neq a_3 \neq a_1$; b) $a_1 = a_2 = a_3$.